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Poisson structure of the Liouville field theory

I M Khamitov
Alkor Technologies Inc, PO Box 116, St Petersburg 199155, Russia

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Abstract. A new analysis of the Poisson structure of the Liouville field theory (LFT) in an infinite volume is presented. The second Poisson structure of the Korteweg–de Vries equation is thoroughly investigated as an essential part of the approach, and a variety of correct Poisson brackets is found. This (along with other causes) leads to a variety of correct LFT Poisson structures. Special attention is paid to the most important LFT property of conformal invariance. In particular, a maximal conformal group suitable for the adopted LFT phase spaces is found, and various Hamiltonian representations of the conformal algebra are described. The properties of local commutativity and canonicity are proved.

1. Introduction

In this paper we consider the Liouville equation

$$\Box\Phi + 4\,e^{2\Phi} = 0 \qquad \Box = \partial^2/\partial t^2 - \partial^2/\partial x^2 \tag{1.1}$$

for a real scalar field $\Phi(t, x)$ as a non-trivial model of the massless field theory in two-dimensional space-time, the so-called Liouville field theory (LFT). In particular, the space-time coordinates vary on the whole real axis, i.e. $t\in\mathbb{R}$, $x\in\mathbb{R}$. The problem of quantization of LFT was discussed in [1–6], but the results achieved in this direction leave much to be desired. The canonical structure of LFT was considered in a number of papers [1, 7, 8]. In this paper we present a new analysis of the LFT Poisson structure in the hope that the one developed here will be helpful for further quantization purposes.

The Hamiltonian description of LFT is *formally* given as follows. First, equation (1.1) must be provided with initial data

$$\varphi(x) = \Phi(t, x)|_{t=0} \qquad \pi(x) = \frac{\partial}{\partial t}\Phi(t, x)|_{t=0}. \tag{1.2}$$

The functions $\varphi(x)$ and $\pi(x)$ are considered as coordinates on the infinite-dimensional phase space of the model. Second, the Poisson structure is defined by the canonical brackets

$$\{\pi(x), \varphi(y)\} = \delta(x-y) \qquad \{\pi(x), \pi(y)\} = \{\varphi(x), \varphi(y)\} = 0. \tag{1.3}$$

Finally, the Hamiltonian

$$H = \int dx\left(\tfrac{1}{2}\pi^2(x) + \tfrac{1}{2}(\varphi'(x))^2 + 2\,e^{2\varphi(x)}\right) \tag{1.4}$$

generates the equation of motion (1.1) in the usual way. This scheme is filled with mathematical meaning in the main body of the paper.

In the next section we shortly discuss the Cauchy problem (1.1), (1.2), essentially following [9], but our smoothness requirement for the Cauchy data is somewhat stronger. In section 3 we specify the boundary conditions and describe the space $\mathscr{M}$ from which various LFT phase spaces are extracted later on in section 7. We also define a bijection, $J^{-1}: (\pi, \varphi) \rightarrow (U_+, U_-, T)$, which provides a partial separation of modes. The functions $U_+(x)$, $U_-(x)$ and the 2×2 matrix T may be regarded as alternative coordinates on the respective phase space. In section 4 we discuss the conformal symmetry of LFT, and in particular, we find a maximal conformal group which acts in our phase spaces.

The canonical brackets (1.3) suggest that the function $U_+(x)$ (or $U_-(x)$) has the Poisson bracket identical with that of the Korteweg–de Vries equation (KdV) [10, 11]. So in section 5 we thoroughly investigate the second KdV Poisson structure which is, however, not fixed by (1.3) uniquely: there is a whole variety of correct Poisson structures.

In section 6 local commutativity and canonicity of the fundamental field is proved. In section 7 Hamiltonian representations of the conformal algebra are described for various phase spaces and various non-degenerate Poisson structures on them.

Our approach to the LFT Poisson structure differs from that of [7, 8] in the following respects:

(1) We consider LFT in 'a general position'. In particular, the LFT phase space adopted in [7, 8] may be regarded as a subspace of some of our phase spaces (its codimension being $\geqslant 2$). In fact, we exclude from our phase spaces a set of codimension 1 which contains the phase space of [7, 8]. Thus, the present paper and [7, 8], so to speak, do not intersect.

(2) We introduce and discuss a whole variety of Poisson structures compatible with the formal formulae (1.3).

(3) We consider both regular and singular solutions to the Liouville equation.

2. Cauchy problem for the Liouville equation

A 'general solution' to the equation (1.1) can be represented in the following form [9]:

$$\Phi(x^+, x^-) = -\log|f_{+1}(x^+)f_{-1}(x^-) + f_{+2}(x^+)f_{-2}(x^-)| \tag{2.1}$$

where the cone variables $x^{\pm} = x \pm t$, $\partial_{\pm} = \frac{1}{2}(\partial/\partial x \pm \partial/\partial t)$ are introduced, and the real-valued functions $f_{\pm k}$ satisfy the condition

$$W(f_{\pm 1}, f_{\pm 2}) \equiv \pm 1 \tag{2.2}$$

($W(f, g)$ is the Wronskian $fg' - f'g$). Formula (2.1) implies an analogous representation for the Cauchy data:

$$\begin{aligned} \varphi(x) &= -\log|f_{+1}(x)f_{-1}(x) + f_{+2}(x)f_{-2}(x)| \\ \pi(x) &= -\frac{f'_{+1}(x)f_{-1}(x) + f'_{+2}(x)f_{-2}(x) - f_{+1}(x)f'_{-1}(x) - f_{+2}(x)f'_{-2}(x)}{f_{+1}(x)f_{-1}(x) + f_{+2}(x)f_{-2}(x)}. \end{aligned} \tag{2.3}$$

The quantities Φ, φ and π remain unchanged if the functions $f_{\pm 1}$, $f_{\pm 2}$ are replaced according to the rule

$$(f_{+1}(x), f_{+2}(x)) \to (f_{+1}(x), f_{+2}(x))T\varepsilon$$
$$\begin{pmatrix} f_{-1}(x) \\ f_{-2}(x) \end{pmatrix} \to T^{-1}\begin{pmatrix} f_{-1}(x) \\ f_{-2}(x) \end{pmatrix} \tag{2.4}$$

where $\varepsilon \in \mathbb{R}$, $\varepsilon^2 = 1$; T is a real x-independent 2×2 matrix with $\det T = 1$.

Smoothness requirement. We consider only those solutions (Cauchy data) to the Liouville equation (1.1) which admit the representation (2.1)–(2.3) with $f_{\pm k} \in C^\infty(\mathbb{R})$, $k = 1, 2$.

The following definition and proposition 2.1 express this requirement explicitly in terms of the Cauchy data.

Definition. Functions $\pi(x)$ and $\varphi(x)$ are said to be proper Cauchy data if there exists a closed set of isolated points $K \subset \mathbb{R}$ (empty, finite or countable) such that $\pi, \varphi \in C^\infty(\mathbb{R} \setminus K)$, and in some neighbourhood of every point $x_i \in K$ the following representation holds:

$$\varphi(x) = -\frac{1}{2}\log\frac{4(x-x_i)^2}{1-v_i^2} + \varphi_i(x) \qquad \pi(x) = v_i\left(\frac{1}{x-x_i} + \varphi_i'(x)\right) + \pi_i(x) \tag{2.5}$$

where $|v_i| < 1$, the functions π_i, φ_i are C^∞-smooth in the neighbourhood of x_i, and $\pi_i(x_i) = \varphi_i(x_i) = 0$.

Proposition 2.1. (a) Let $f_{\pm 1}, f_{\pm 2} \in C^\infty(\mathbb{R})$ and the condition (2.2) be satisfied, then the Cauchy data given by the formulae (2.3) are proper.

(b) Let the Cauchy data $\pi(x)$, $\varphi(x)$ be proper, then there exist, unique up to transformation (2.4), functions $f_{\pm 1}, f_{\pm 2} \in C^\infty(\mathbb{R})$ satisfying the condition (2.2) and such that the representation (2.3) holds for $\pi(x)$ and $\varphi(x)$.

This proposition is in fact equivalent to the unique existence theorem for the Cauchy problem (1.1), (1.2) with proper Cauchy data, because the same functions $f_{\pm k}$ participate in formula (2.1) for Φ. Singular points of the Cauchy data generate singular curves of the solution Φ, which are time-like and do not intersect each other. In particular, non-singular Cauchy data generate a non-singular (smooth) solution to the Liouville equation [9, 12, 13].

3. Boundary condition and the space $\mathscr{M}$

It follows from the equation of motion (1.1) that the functions $U_\pm = (\partial_\pm \Phi)^2 - \partial_\pm^2 \Phi$ depend only on one variable, x^+ or x^-: $U_+ = U_+(x^+)$, $U_- = U_-(x^-)$. Hence, they can be expressed in terms of the Cauchy data,

$$U_\pm(x) = \left(\frac{\varphi'(x) \pm \pi(x)}{2}\right)^2 - \left(\frac{\varphi'(x) \pm \pi(x)}{2}\right)' + e^{2\varphi(x)} \tag{3.1}$$

or the functions $f_{\pm k}$ [9, 12, 13],

$$-f''_{\pm k}(x)+U_{\pm}(x)f_{\pm k}(x)=0 \qquad k=1,2. \tag{3.2}$$

Due to our smoothness requirement, $U_{\pm}\in C^{\infty}(\mathbb{R})$. Now we impose the boundary condition on the fields involved by requiring rapid decrease of $U_{\pm}$:

$$U_{\pm}\in S(\mathbb{R}) \tag{3.3}$$

where $S(\mathbb{R})$ is a Schwartz space of C^{∞}-smooth real functions of $x\in\mathbb{R}$, which decrease with all their derivatives faster than any power of x^{-1} when $|x|\to\infty$. In proposition 3.1 we shall make this boundary condition explicit, but to formulate it we need to introduce a useful notation similar to $o(x)$. Let $\eta(x)$ be a C^{∞}-function such that $\eta(x)=0$ if $x<0$, and $\eta(x)=1$ if $x>1$. Then for $f\in C^{\infty}(\mathbb{R})$, we write $f(x)=s_+(x)$ if $\eta f\in S(\mathbb{R})$, and $f(x)=s_-(x)$ if $f(-x)=s_+(x)$. For example, $\tanh x=1+s_+(x)=-1+s_-(x)$.

Proposition 3.1. Let π and φ be proper Cauchy data, then (3.3) is equivalent to one of the following four types of boundary behaviour of π and φ:

(a)
$$\begin{aligned}\pi(x)&=\frac{\tanh u_{\pm}}{x-q_{\pm}}+s_{\pm}(x)\\ \varphi(x)&=-\log(2|x-q_{\pm}|\cosh u_{\pm})+s_{\pm}(x)\end{aligned} \tag{3.4a}$$

(b)
$$\begin{aligned}\pi(x)&=\frac{-2\tau_+}{(x-q_+)^2-\tau_+^2-1/B_+^2}+s_+(x)=\frac{\tanh u_-}{x-q_-}+s_-(x)\\ \varphi(x)&=-\log(B_+|(x-q_+)^2-\tau_+^2-1/B_+^2|)+s_+(x)\\ &=-\log(2|x-q_-|\cosh u_-)+s_-(x)\end{aligned} \tag{3.4b}$$

(c)
$$\begin{aligned}\pi(x)&=\frac{\tanh u_+}{x-q_+}+s_+(x)=\frac{-2\tau_-}{(x-q_-)^2-\tau_-^2-1/B_-^2}+s_-(x)\\ \varphi(x)&=-\log(2|x-q_+|\cosh u_+)+s_+(x)\\ &=-\log(B_-|(x-q_-)^2-\tau_-^2-1/B_-^2|)+s_-(x)\end{aligned} \tag{3.4c}$$

(d)
$$\begin{aligned}\pi(x)&=\frac{-2\tau_{\pm}}{(x-q_{\pm})^2-\tau_{\pm}^2-1/B_{\pm}^2}+s_{\pm}(x)\\ \varphi(x)&=-\log(B_{\pm}|(x-q_{\pm})^2-\tau_{\pm}^2-1/B_{\pm}^2|)+s_{\pm}(x).\end{aligned} \tag{3.4d}$$

Here $B_{\pm}>0$ and the constants $u_{\pm}$, $q_{\pm}$, $\tau_{\pm}$ can take arbitrary values. The corresponding solution Φ has the following asymptotic behaviour in each case:

(a) $$\Phi(t,x)=-\log(2|(x-q_{\pm})\cosh u_{\pm}-t\sinh u_{\pm}|)+s_{\pm}(x)$$

(b) $$\begin{aligned}\Phi(t,x)&=-\log(B_+|(x-q_+)^2-(t-\tau_+)^2-1/B_+^2|)+s_+(x)\\ &=-\log(2|(x-q_-)\cosh u_--t\sinh u_-|)+s_-(x)\end{aligned}$$

(c) $$\begin{aligned}\Phi(t,x)&=-\log(2|(x-q_+)\cosh u_+-t\sinh u_+|)+s_+(x)\\ &=-\log(B_-|(x-q_-)^2-(t-\tau_-)^2-1/B_-^2|)+s_-(x)\end{aligned}$$

(d) $$\Phi(t,x)=-\log(B_{\pm}|(x-q_{\pm})^2-(t-\tau_{\pm})^2-1/B_{\pm}^2|)+s_{\pm}(x).$$

The proper Cauchy data obeying the boundary condition (3.3), or equivalently (3.4), constitute a space which seems to be an appropriate candidate for the phase space of the model, but we want to exclude from this space some sets of codimension 1, namely we require that both Schrödinger equations (3.2) should not have a virtual eigenvalue (the opposite was assumed in [7, 8]). To make this requirement more exact and introduce some notations we are going to discuss the one-dimensional Schrödinger equation in more detail.

The equation $-f''(x)+U(x)f(x)=0$ with $U\in S(\mathbb{R})$ has solutions exhibiting the following asymptotic behaviour:

$$\begin{aligned}\psi_1(x)&=1+s_-(x)=\alpha+\beta x+s_+(x)\\ \psi_2(x)&=x+s_-(x)=\gamma+\vartheta x+s_+(x)\\ \chi_1(x)&=1+s_+(x)=\vartheta-\beta x+s_-(x)\\ \chi_2(x)&=x+s_+(x)=-\gamma+\alpha x+s_-(x)\end{aligned} \tag{3.5}$$

where

$$\alpha\vartheta-\beta\gamma=1. \tag{3.6}$$

Let us define some subsets of $S(\mathbb{R})$ $(n=0, 1, 2, \dots)$:

$$M_n=\{U|\, U\in S(\mathbb{R}),\ \psi_1^U \text{ is unbounded and has exactly } n \text{ zeros}\}$$

$$D_n=\{U|\, U\in S(\mathbb{R}),\ \psi_1^U \text{ is bounded and has exactly } n \text{ zeros}\}.$$

The standard locally convex topology on $S(\mathbb{R})$ [14] is meant in the following proposition which collects some simple properties of the sets M_n, D_n.

Proposition 3.2.

(1) $\left(\bigcup_{n=0}^{\infty} M_n\right)\cup\left(\bigcup_{n=0}^{\infty} D_n\right)=S(\mathbb{R})$, $M_n\cap D_m=\phi$, $M_n\cap M_m=\phi$ if $n\neq m$, $D_n\cap D_m=\phi$ if $n\neq m$.

(2) M_n is open and simply connected.

(3) D_n is closed, simply connected, and codim $D_n=1$.

(4) $D_n\cup D_{n-1}$ is a boundary of M_n $(D_{-1}=\phi)$, i.e. $D_n\cup D_{n-1}=$ closure of M_n minus M_n.

(5) D_n separates M_n and M_{n+1}, i.e. any continuous curve $w\colon [0, 1]\to S(\mathbb{R})$ beginning in M_n $(w(0)\in M_n)$ and ending in M_{n+1} $(w(1)\in M_{n+1})$, intersects D_n at least one time.

For the potential

$$U\in\bigcup_{n=0}^{\infty} M_n$$

we can define another two solutions to the Schrödinger equation by the formulae

$$\psi(x)=|\beta|^{-1/2}\psi_1(x) \qquad \chi(x)=|\beta|^{-1/2}\chi_1(x). \tag{3.7}$$

Note that if $U\in M_n$, then sign $\beta=(-1)^n=W(\chi, \psi)$.

Let us consider a set

$$\mathscr{M}=\bigcup_{\substack{n_+=0\\n_-=0}}^{\infty} M_{n_+}\times M_{n_-}\times PSL_2 \tag{3.8}$$

where $PSL_2=SL_2(\mathbb{R})/\{\pm\mathrm{id}\}$. $\mathscr{M}$ is an open and disconnected subset of $S(\mathbb{R})\times S(\mathbb{R})\times PSL_2$, the triples (U_+, U_-, T) being its points. The phase space of LFT will be identified in section 7 with some subsets of $\mathscr{M}$, which will be defined by some subsets of PSL_2. Such a reduction of $\mathscr{M}$ is necessary to obtain non-degenerate Poisson brackets.

The formula (3.8) must be accompanied with an injective mapping

$$J: (U_+, U_-, T)\to(\pi,\varphi) \tag{3.9}$$

which enables the Cauchy data π, φ to play a role of alternative coordinates on $\mathscr{M}$. To define J we only need to specify particular $f_{\pm 1}$ and $f_{\pm 2}$ in (2.2), (2.3) provided that U_+, U_- and T are given. Define ($U_\pm\in M_{n_\pm}$)

$$\begin{aligned}&(f_{+1}(x), f_{+2}(x))=(\chi_+(x), (-1)^{n_+}\psi_+(x))\equiv\Omega_+(x)\\&\begin{pmatrix}f_{-1}(x)\\f_{-2}(x)\end{pmatrix}=T\begin{pmatrix}(-1)^{n_-}\psi_-(x)\\\chi_-(x)\end{pmatrix}\equiv T\Omega_-(x)\end{aligned} \tag{3.10}$$

then the representations for Φ (2.1) and π, φ (2.3) take the form

$$\Phi(x^+, x^-)=-\log|\Omega_+(x^+)T\Omega_-(x^-)| \tag{3.11}$$

$$\begin{cases}\varphi(x)=-\log|\Omega_+(x)T\Omega_-(x)|\\ \pi(x)=-\dfrac{\Omega'_+(x)T\Omega_-(x)-\Omega_+(x)T\Omega'_-(x)}{\Omega_+(x)T\Omega_-(x)}.\end{cases} \tag{3.12}$$

The formulae (3.10), (3.12) give a desired definition of the mapping J (3.9) which appears to be an injection, as can be learned from proposition 2.1.

Proposition 3.3. Let

$$U_+\in M_{n_+}\qquad U_-\in M_{n_-}\qquad T=\begin{pmatrix}a&b\\c&d\end{pmatrix}\qquad ad-bc=1$$

and the corresponding Cauchy data π, φ have exactly n points of singularity, then:

(a) $a, d>0, b, c\geqslant 0$ ($a, d<0, b, c\leqslant 0$) $\Leftrightarrow n=n_++n_-$;
(b) $a, d>0, b, c<0$ ($a, d<0, b, c>0$) $\Leftrightarrow n=n_++n_-+2$;
(c) all other cases $\Leftrightarrow n=n_++n_-+1$.

Note also that if π and φ are nonsingular, then $U_\pm\in M_0$, in particular, the virtual eigenvalues are automatically absent.

Proposition 3.4. The Cauchy data exhibit the following boundary behaviour depending upon

$$T=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

(a) $b=0,\ c=0 \Leftrightarrow (3.4a)$;
(b) $b=0,\ c\neq 0 \Leftrightarrow (3.4b)$;
(c) $b\neq 0,\ c=0 \Leftrightarrow (3.4c)$;
(d) $b\neq 0,\ c\neq 0 \Leftrightarrow (3.4d)$.

We show in the appendix the expressions for $u_\pm$, $q_\pm$, $\tau_\pm$, $B_\pm$ via T, $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$ and $\vartheta_\pm$ in each case. These expressions permit the matrix T to be extracted from the given π and φ, thereby, together with (3.1), defining the mapping inverse to (3.9).

4. Conformal symmetry

The conformal group $\mathrm{Conf}(\mathbb{M}^2)=\mathrm{Diff}_+^\infty(\mathbb{R})\times\mathrm{Diff}_+^\infty(\mathbb{R})$ of the two-dimensional Minkowski space $\mathbb{M}^2$ consists of mappings $F\colon \mathbb{M}^2\to\mathbb{M}^2$ which have the following form:

$$x^+\to y^+=F^+(x^+)\qquad x^-\to y^-=F^-(x^-)$$

where $F^\pm$ are arbitrary orientation-preserving C^∞-diffeomorphisms of $\mathbb{R}$. The corresponding conformal algebra $\mathrm{conf}(\mathbb{M}^2)=\mathrm{Vect}^\infty(\mathbb{R})\oplus\mathrm{Vect}^\infty(\mathbb{R})$ consists of vector fields X on $\mathbb{M}^2$ which have the form

$$X=X^+(x^+)\partial_+ + X^-(x^-)\partial_-\qquad X^\pm\in C^\infty(\mathbb{R}).$$

The Liouville equation (1.1) is invariant against the conformal transformations, provided they act on $\Phi(x^+,x^-)$ as follows $(F=F^+\times F^-\in\mathrm{Conf}(\mathbb{M}^2))$:

$$\Phi^F(x^+,x^-)=\Phi(F^+(x^+),F^-(x^-))+\tfrac{1}{2}\log(\partial F^+(x^+)\,\partial F^-(x^-)).$$

The potential $U_\pm$ is non-trivially transformed only by the component $F^\pm$ of the element $F^+\times F^-\in\mathrm{Conf}(\mathbb{M}^2)$. The formulae are the same for both signs, so in what follows we occasionally omit the '$\pm$' indices. For $F\in\mathrm{Diff}_+^\infty(\mathbb{R})$, we have

$$U^F(x)=U(F(x))(F'(x))^2+\left(\frac{F''(x)}{2F'(x)}\right)^2-\left(\frac{F''(x)}{2F'(x)}\right)'. \tag{4.1}$$

The group $\mathrm{Conf}(\mathbb{M}^2)$ and the algebra $\mathrm{conf}(\mathbb{M}^2)$ are too big to have physical meaning. Appropriate for our purposes is the conformal subgroup $\mathfrak{G}=\mathscr{D}\times\mathscr{D}\subset\mathrm{Conf}(\mathbb{M}^2)$, where

$$\mathscr{D}=\{F\mid F\in\mathrm{Diff}_+^\infty(\mathbb{R}),\ F''\in S(\mathbb{R})\}\subset\mathrm{Diff}_+^\infty(\mathbb{R}).$$

We shall also make use of the subgroup $\mathfrak{G}_*=\mathscr{D}_*\times\mathscr{D}_*\subset\mathfrak{G}$, where

$$\mathscr{D}_*=\{F\mid F\in\mathrm{Diff}_+^\infty(\mathbb{R}),\ F''\in S(\mathbb{R}),\ F'(+)=F'(-)\}\qquad F'(\pm)=\lim_{x\to\pm\infty}F'(x).$$

The respective conformal subalgebras are $\mathscr{A}_*=\mathscr{V}_*\oplus\mathscr{V}_*\subset\mathscr{A}=\mathscr{V}\oplus\mathscr{V}\subset\mathrm{conf}(\mathbb{M}^2)$, where

$$\mathscr{V}=\left\{X\mid X=X(x)\frac{\mathrm{d}}{\mathrm{d}x},\ X''(\cdot)\in S(\mathbb{R})\right\}$$

$$\mathscr{V}_*=\left\{X\mid X=X(x)\frac{\mathrm{d}}{\mathrm{d}x},\ X''(\cdot)\in S(\mathbb{R}),\ X'(+)=X'(-)\right\}.$$

The algebra $\mathscr{V}$ has two non-trivial 2-cocycles:

$$\omega(X, Y)=\tfrac{1}{2}\int \mathrm{d}x(X''(x)Y'(x)-X'(x)Y''(x)) \tag{4.2}$$

$$\tilde{\omega}(X, Y)=X'(+)Y'(-)-Y'(+)X'(-). \tag{4.3}$$

The cocycle $\tilde{\omega}$ vanishes on the subalgebra $\mathscr{V}_*$.

Both the algebras $\mathscr{A}$ and $\mathscr{A}_*$ are satisfactory from the physical point of view because they include the generators of time translations ($X^+(x)\equiv 1, X^-(x)\equiv -1$), space translations ($X^+(x)\equiv 1$, $X^-(x)\equiv 1$), Lorentz rotations ($X^+(x)=x, X^-(x)=-x$), and dilatations ($X^+(x)=x$, $X^-(x)=x$). Correspondingly, the conformal groups $\mathfrak{G}$ and $\mathfrak{G}_*$ include space-time translations, Lorentz rotations, and dilatations.

It is obvious from (4.1) that if $U\in S(\mathbb{R})$ and $F\in\mathscr{D}$, then $U^F\in S(\mathbb{R})$. The converse of this statement is also true:

Proposition 4.1. Let F be an orientation-preserving C^3-diffeomorphism of $\mathbb{R}$ and $U\in S(\mathbb{R})$. If $U^F\in S(\mathbb{R})$, then $F\in\mathscr{D}$.

Thus, $\mathscr{D}$ is a maximal group of the diffeomorphisms acting in $S(\mathbb{R})$ by the formula (4.1), correspondingly, $\mathfrak{G}$ is a maximal conformal group suitable for our choice of the phase spaces.

Proposition 4.2. (a) The sets $M_n, D_n, n=0, 1, 2, \ldots$ are orbits of the group $\mathscr{D}$ with respect to its action (4.1) in $S(\mathbb{R})$.

(b) The sets $M_n, n=0, 1, 2, \ldots$ are orbits of the group $\mathscr{D}_*$ with respect to its action (4.1) in $S(\mathbb{R})$; and for any U, $V\in M_n$, there exists one and only one diffeomorphism $F\in\mathscr{D}_*$ such that $U^F=V$.

(c) The set D_n ($n=0, 1, 2, \ldots$) is a union of the orbits of the group $\mathscr{D}_*$, each of them being distinguished by the additional condition $\alpha=$constant (if $U\in D_n$, then $\alpha\neq 0$, sign $\alpha=(-1)^n$).

This proposition, in particular, means that only the whole sets M_n (not their proper subsets) can participate in the construction of the phase space of the model to preserve the conformal invariance.

It follows directly from (4.1) that we have for $F\in\mathscr{D}$

$$\beta^F=\beta(F'(+)F'(-))^{1/2} \tag{4.4}$$

$$\psi^F(x)=\psi(F(x))(F'(x))^{-1/2}(F'(+)/F'(-))^{-1/4} \tag{4.5}$$

$$\chi^F(x)=\chi(F(x))(F'(x))^{-1/2}(F'(+)/F'(-))^{1/4}. \tag{4.6}$$

The response of the matrix T to the conformal transformations can be obtained from the condition of commutativity of the diagram:

$$\begin{array}{ccc} (U_+, U_-, T) & \xrightarrow{J} & (\pi, \varphi) \sim \Phi \\ \Big\downarrow {\scriptstyle F=F^+\times F^-} & & \Big\downarrow {\scriptstyle F=F^+\times F^-} \\ (U_+^F, U_-^F, T^F) & \xrightarrow{J} & (\pi^F, \varphi^F) \sim \Phi^F \end{array}$$

The result is:

$$T^F = \begin{pmatrix} 1/r_+ & 0 \\ 0 & r_+ \end{pmatrix} T \begin{pmatrix} 1/r_- & 0 \\ 0 & r_- \end{pmatrix}^{-1} \qquad r_\pm = \left(\frac{\partial F^\pm(+)}{\partial F^\pm(-)} \right)^{1/4}. \tag{4.7}$$

5. Second KdV Poisson structure

Formal application of the canonical brackets (1.3) to the potentials $U_\pm$ (3.1) yields

$$\{U_+(x), U_+(y)\} = -(U_+(x) + U_+(y))\delta'(x-y) + \tfrac{1}{2}\delta'''(x-y) \tag{5.1}$$

$$\{U_-(x), U_-(y)\} = (U_-(x) + U_-(y))\delta'(x-y) - \tfrac{1}{2}\delta'''(x-y) \tag{5.2}$$

$$\{U_+(x), U_-(y)\} = 0. \tag{5.3}$$

In this section we investigate the bracket (5.1) keeping in mind that the bracket (5.2) can be obtained from (5.1) by a mere reflection of the total sign. For this reason, we omit the index '+' in the notation of U_+ and in the notation of other quantities which inherit it from U_+. The bracket (5.1) is known as the second one in the hierarchy of KdV brackets [10, 11]. The overall Poisson structure will be described in the next two sections.

Let $d_V f$ be a Gâteaux derivative of a C^1-class functional $f\colon M \to \mathbb{R}$ which is defined on an open set $M \subset S(\mathbb{R})$. It is *linear* and *continuous* with respect to $V \in S(\mathbb{R})$ [15], so it can be written in the integral form:

$$d_V f = \int dx\, V(x) \frac{\delta f}{\delta U(x)}$$

which is to be regarded as a definition of the variational derivative $\delta f/\delta U(x)$. Let us introduce another two differential operators

$$d_\pm f = \lim_{x\to\pm\infty} (\partial/\partial x)(\delta f/\delta U(x))$$

in those cases when this limit makes sense and exists. For example, this is the case if $(\partial^2/\partial x^2)(\delta f/\delta U(x)) \in S(\mathbb{R})$ as a function of x.

Let us define the algebra $\mathcal{O}(M)$ of admissible functionals (observables) f defined on an open set $M \subset S(\mathbb{R})$ by requiring the fulfilment of the following recursive conditions:

(1) f is a C^1-functional with respect to the variable $U \in M$; further, $(\partial^2/\delta x^2)(\delta f/\delta U(x)) \in S(\mathbb{R})$ as a function of x, and the condition 2 is fulfilled;

(2) the functionals $d_+ f$, $d_- f$ and $d_V f$, $V \in S(\mathbb{R})$ satisfy the condition 1, and $(d_\pm d_V - d_V d_\pm) f = 0$, $\forall V \in S(\mathbb{R})$.

The operators d_+ and d_- do not commute as the following example demonstrates. Let

$$f(U)=\tfrac{1}{2}\iint dx\,dy\,U(x)U(y)xy\tanh(x+y)$$

then $f\in\mathcal{O}(M)$ for any open M, and $d_+d_-f=-d_-d_+f=1$ (compare [16]). However, no one of the functionals essential to our theory shares this property, therefore, one can reduce the algebra of admissible functionals to the algebra $\mathcal{O}'(M)\subset\mathcal{O}(M)$ by imposing an additional condition

$$(d_+^k d_-^l - d_-^l d_+^k)f=0 \qquad k,l=1,2,3,\ldots$$

or by inserting the condition $(d_+d_- - d_-d_+)f=0$ into item 2 of the above recursive definition. As a matter of fact further reduction to the algebra $\mathcal{O}''(M)\subset\mathcal{O}'(M)$, whose elements satisfy the condition $(d_+^k - d_-^k)f=0$, $k=1,2,3,\ldots$ is possible (see the remark after proposition 5.1). Alternatively, the algebra $\mathcal{O}''(M)$ may be defined by inserting the condition $(d_+ - d_-)f=0$ into item 2 of the above recursive definition.

Let us examine whether the functionals of U which were introduced by (3.5), (3.7) are admissible. It is easy to verify that the functionals $\psi_1(x)$, $\chi_1(x)$, $\psi_2(x)$, and $\chi_2(x)$ are those of C^1-class and their variational derivatives look as follows $(i=1,2)$:

$$\frac{\delta\psi_i(x)}{\delta U(y)}=-\theta(x-y)\psi_i(y)[\psi_1(x)\psi_2(y)-\psi_2(x)\psi_1(y)] \tag{5.4}$$

$$\frac{\delta\chi_i(x)}{\delta U(y)}=\theta(y-x)\chi_i(y)[\chi_1(x)\chi_2(y)-\chi_2(x)\chi_1(y)] \tag{5.5}$$

where $\theta(x)$ is a step function: $\theta(x)=1$ if $x\geqslant 0$, and $\theta(x)=0$ if $x<0$. It follows from the formulae $\alpha=W(\psi_1,\chi_2)$, $\beta=W(\chi_1,\psi_1)$, $\gamma=W(\psi_2,\chi_2)$, and $\vartheta=W(\chi_1,\psi_2)$ that the functionals α, β, γ, and ϑ are of C^1-class too, their variational derivatives being

$$\begin{aligned}\frac{\delta\beta}{\delta U(y)}&=\chi_1(y)\psi_1(y) \qquad & \frac{\delta\alpha}{\delta U(y)}&=-\chi_2(y)\psi_1(y)\\ \frac{\delta\gamma}{\delta U(y)}&=-\chi_2(y)\psi_2(y) \qquad & \frac{\delta\vartheta}{\delta U(y)}&=\chi_1(y)\psi_2(y).\end{aligned} \tag{5.6}$$

It immediately follows from these formulae that the functionals $\psi_2(x)$, $\chi_2(x)$, α, γ, and ϑ are not admissible, because their variational derivatives grow quadratically when $|y|\to\infty$.

Proposition 5.1. β, $\psi_1(\xi)$, $\chi_1(\xi)\in\mathcal{O}'(M)$ and $\psi(\xi)$, $\chi(\xi)\in\mathcal{O}''(M)$ for any open set

$$M\subset\bigcup_{n=0}^{\infty}M_n$$

and any $\xi\in S(\mathbb{R})$, where $\psi_1(\xi)=\int dx\,\xi(x)\psi_1(x)$, etc.

Note that only $\psi(x)$ and $\chi(x)$ participate in the formulae (3.11), (3.12) for the fields π, φ and Φ. Let us give a sketch of the proof, omitting smoothing for the sake of brevity. Rewrite the formulae (5.4) and (5.5) for $i=1$ as follows:

$$\frac{\delta\psi_1(x)}{\delta U(y)}=-\theta(x-y)\psi_1(y)\frac{1}{\beta}[\chi_1(x)\psi_1(y)-\psi_1(x)\chi_1(y)] \tag{5.7}$$

$$\frac{\delta\chi_1(x)}{\delta U(y)}=\theta(y-x)\chi_1(y)\frac{1}{\beta}[\chi_1(x)\psi_1(y)-\psi_1(x)\chi_1(y)] \tag{5.8}$$

and complement them with their limiting forms:

$$\begin{array}{lll} d_+\beta=\beta & d_+\psi_1(x)=0 & d_+\chi_1(x)=\chi_1(x) \\ d_-\beta=-\beta & d_-\psi_1(x)=-\psi_1(x) & d_-\chi_1(x)=0. \end{array} \tag{5.9}$$

The formulae (5.6)–(5.9) show that various derivatives of the functionals $1/\beta$, $\psi_1(x)$ and $\chi_1(x)$ are *polynomially* expressed via the same functionals. It is readily seen from (5.9) that the differential operator of the *first* order $d_+d_- - d_-d_+$ annihilates the functionals $1/\beta$, $\psi_1(x)$ and $\chi_1(x)$, and, due to the polynomial character of the formulae (5.6)–(5.9), all their derivatives, as is required by the recursive definition. It is analogously verified that the differential operators of the *first* order $d_\pm d_\nu - d_\nu d_\pm$ annihilate $1/\beta$, $\psi_1(x)$, $\chi_1(x)$ and all their derivatives. This proves the proposition in the part concerning the functionals β, $\psi_1(x)$ and $\chi_1(x)$. The rest of the proposition is analogously proved with the help of the following formulae ($U\in M_n$):

$$(-1)^n\frac{\delta\psi(x)}{\delta U(y)}=-\tfrac{1}{2}\chi(y)\psi(y)\psi(x)-\theta(x-y)\psi(y)[\chi(x)\psi(y)-\psi(x)\chi(y)]$$

$$(-1)^n\frac{\delta\chi(x)}{\delta U(y)}=-\tfrac{1}{2}\chi(y)\psi(y)\chi(x)+\theta(y-x)\chi(y)[\chi(x)\psi(y)-\psi(x)\chi(y)]$$

$$d_+\psi(x)=d_-\psi(x)=-\tfrac{1}{2}\psi(x) \qquad d_+\chi(x)=d_-\chi(x)=\tfrac{1}{2}\chi(x).$$

The bracket (5.1) is a formal consequence of the following bracket which is defined on the algebras of admissible functionals $\mathcal{O}(M)$, $\mathcal{O}'(M)$ and $\mathcal{O}''(M)$:

$$\{f,g\}=(f,g)+\langle f,g\rangle+C\langle\!\langle f,g\rangle\!\rangle \tag{5.10}$$

where $C\in\mathbb{R}$ and

$$(f,g)=-\int dx\, U(x)\left[\frac{\delta f}{\delta U(x)}\left(\frac{\delta g}{\delta U(x)}\right)'-\left(\frac{\delta f}{\delta U(x)}\right)'\frac{\delta g}{\delta U(x)}\right]$$

$$\langle f,g\rangle=\tfrac{1}{4}\int dx\left[\left(\frac{\delta f}{\delta U(x)}\right)''\left(\frac{\delta g}{\delta U(x)}\right)'-\left(\frac{\delta f}{\delta U(x)}\right)'\left(\frac{\delta g}{\delta U(x)}\right)''\right]$$

$$\langle\!\langle f,g\rangle\!\rangle=d_+f\,d_-g-d_-f\,d_+g.$$

The last term in (5.10) vanishes on the algebra $\mathcal{O}''(M)$.

It is quite obvious that the bracket (5.10) is bilinear, antisymmetric and satisfies the Leibniz identity. Not obvious are the Jacobi identity and the properties of degeneracy.

Proposition 5.2. (a) The bracket (5.10) satisfies the Jacobi identity in the case of the algebra $\mathcal{O}(M)$ if and only if $C=\frac{1}{4}$ or $C=-\frac{1}{4}$. (b) The bracket (5.10) satisfies the Jacobi identity in the case of the algebra $\mathcal{O}'(M)$ for any $C\in\mathbb{R}$.

The proposition 5.2 is a consequence of the following equality (compare this with the case of the first KdV Poisson structure [16, 17]):

$$\{\{f, g\}, h\} + \text{cycle} = (1/16 - C^2)\langle\langle f, g\rangle\rangle(\mathrm{d}_+\mathrm{d}_- - \mathrm{d}_-\mathrm{d}_+)h + \text{cycle}.$$

Proposition 5.3. (1) Let the algebra of admissible functionals be $\mathcal{O}''$. Then the Poisson bracket (5.10) is non-degenerate at every point of the open set

$$\bigcup_{n=0}^{\infty} M_n$$

and the degree of degeneracy is equal to 2 at every point of the complementary set

$$\bigcup_{n=0}^{\infty} D_n.$$

(2) Let the algebra of admissible functionals be $\mathcal{O}$ or $\mathcal{O}'$.

(a) If $C=\frac{1}{4}$, then the Poisson bracket (5.10) is degenerate at every point of $S(\mathbb{R})$, the degree of degeneracy being equal to 1. The functional β is an annihilator (central element) non-trivial at every point of $S(\mathbb{R})$.

(b) If $C=-\frac{1}{4}$, then the Poisson bracket (5.10) is non-degenerate at every point of the open set

$$\bigcup_{n=0}^{\infty} M_n$$

and the degree of degeneracy is equal to 2 at every point of the complementary set

$$\bigcup_{n=0}^{\infty} D_n.$$

(c) If $C^2 \neq \frac{1}{16}$, then the Poisson bracket (5.10) is non-degenerate at every point of the open set

$$\bigcup_{n=0}^{\infty} M_n$$

and the degree of degeneracy is equal to 1 at every point of the complementary set

$$\bigcup_{n=0}^{\infty} D_n.$$

To prove this proposition let us first rewrite the Poisson bracket (5.10) as follows:

$$\{f, g\} = \int \mathrm{d}x \frac{\delta f}{\delta U(x)} \Lambda \frac{\delta g}{\delta U(x)} + \mathrm{d}_+ f(\tfrac{1}{4}\,\mathrm{d}_+ g + C\,\mathrm{d}_- g) - \mathrm{d}_- f(\tfrac{1}{4}\,\mathrm{d}_- g + C\,\mathrm{d}_+ g) \tag{5.11}$$

where

$$\Lambda = \tfrac{1}{2}\frac{\mathrm{d}^3}{\mathrm{d}x^3} - 2U(x)\frac{\mathrm{d}}{\mathrm{d}x} - U'(x). \tag{5.12}$$

It follows from (5.11) that if g is an annihilator of the bracket (5.10) at the point U (i.e. $\delta g/\delta U(x)|_U \not\equiv 0$ as a function of x, and $\{f, g\}|_U = 0$ for any admissible functional f), then $\delta g/\delta U(x)$ is a solution to the equation

$$\Lambda(\delta g/\delta U(x)) = 0 \tag{5.13}$$

and satisfies one of the following boundary conditions:

(a) $\quad d_+ g = -d_- g$ (algebras $\mathcal{O}$, $\mathcal{O}'$ with $C = \frac{1}{4}$) (5.14*a*)

(b) $\quad d_+ g = d_- g$ ($\mathcal{O}$, $\mathcal{O}'$ with $C = -\frac{1}{4}$ or $\mathcal{O}''$) (5.14*b*)

(c) $\quad d_+ g = 0, d_- g = 0$ ($\mathcal{O}$, $\mathcal{O}'$ with $C^2 \neq \frac{1}{16}$). (5.14*c*)

Let us introduce, in the space of solutions to the equation $\Lambda y(x) = 0$, two bases, $\{Y_0^-, Y_1^-, Y_2^-\}$ and $\{Y_0^+, Y_1^+, Y_2^+\}$, which are fixed by the boundary conditions

$$Y_j^\pm(x) = x^j + s_\pm(x) \qquad j = 0, 1, 2$$

and an x-independent transition matrix Z

$$Y_j^+(x) = \sum_{k=0}^{2} Y_k^-(x) Z_j^k.$$

They have the explicit representation

$$Y_0^-(x) = \psi_1^2(x) \qquad Y_1^-(x) = \psi_1(x)\psi_2(x) \qquad Y_2^-(x) = \psi_2^2(x)$$

$$Y_0^+(x) = \chi_1^2(x) \qquad Y_1^+(x) = \chi_1(x)\chi_2(x) \qquad Y_2^+(x) = \chi_2^2(x)$$

$$Z = \begin{pmatrix} \vartheta^2 & -\gamma\vartheta & \gamma^2 \\ -2\beta\vartheta & 2\beta\gamma + 1 & -2\alpha\gamma \\ \beta^2 & -\alpha\beta & \alpha^2 \end{pmatrix}$$

which is a consequence of the following simple proposition.

Proposition 5.4. If $y_1(x)$ and $y_2(x)$ are arbitrary solutions to the equation $-y'' + U(x)y = 0$, then $Y(x) = y_1(x)y_2(x)$ is a solution to the equation $\Lambda Y(x) = 0$, where Λ is given by (5.12).

Resolve a solution to the equation (5.13) with respect to both bases $\{Y_j^+\}$ and $\{Y_j^-\}$:

$$\frac{\delta g}{\delta U(x)} = \sum_{j=0}^{2} y_j^+ Y_j^+(x) = \sum_{j=0}^{2} y_j^- Y_j^-(x)$$
$$(y_0^-, y_1^-, y_2^-)^{\mathrm{T}} = (y_0^+, y_1^+, y_2^+)^{\mathrm{T}} Z^{\mathrm{T}}. \tag{5.15}$$

Consider the boundary condition (5.14*a*). It is equivalent to the equalities $y_2^\pm = 0$, $y_1^- = -y_1^+$, therefore, the equality (5.15) turns into a system of linear equations to determine two coefficients, y_0^+ and y_1^+:

$$\begin{pmatrix} -2\beta\vartheta & 2\beta\gamma + 2 \\ \beta^2 & -\alpha\beta \end{pmatrix} \cdot \begin{pmatrix} y_0^+ \\ y_1^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Taking into account (3.6), we see that for any U, the rank of this system is equal to 1, and a one-dimensional space of solutions is generated by the column $(y_0^+, y_1^+)^{\mathrm{T}} = (\alpha, \beta)^{\mathrm{T}}$. Thus, we arrive at the conclusion that the equation (5.13) + (5.14a) always has only one independent solution for any $U \in S(\mathbb{R})$:

$$\frac{\delta g}{\delta U(x)} = \alpha Y_0^+(x) + \beta Y_1^+(x) = \alpha \chi_1^2(x) + \beta \chi_1(x)\chi_2(x) = \chi_1(x)\psi_1(x).$$

Comparing this with (5.6) we see that β can be chosen as an annihilator. The other assertions of the proposition 5.3 can be proved in the same way.

The property of the operator Λ described in the proposition 5.4 makes it possible to evaluate the Poisson brackets involving one of the functionals β, $\psi_1(x)$, $\chi_1(x)$, $\psi(x)$ or $\chi(x)$. For instance, it immediately follows from (5.6), (5.9) and (5.11) that for any admissible functional f

$$\{f, \beta\} = (\tfrac{1}{4} - C)(\mathrm{d}_+ f + \mathrm{d}_- f)\beta. \tag{5.16}$$

This bracket may be useful in proving that the sets M_n and D_n are closed against the Hamiltonian flows. Similarly, for any admissible functional f, we have

$$\{f, \psi(x)\} = \psi'(x)\frac{\delta f}{\delta U(x)} - \frac{1}{2}\psi(x)\left(\frac{\delta f}{\delta U(x)}\right)' - \frac{1}{2}(C + \tfrac{1}{4})\psi(x)(\mathrm{d}_+ f - \mathrm{d}_- f) \tag{5.17}$$

$$\{f, \chi(x)\} = \chi'(x)\frac{\delta f}{\delta U(x)} - \frac{1}{2}\chi(x)\left(\frac{\delta f}{\delta U(x)}\right)' + \frac{1}{2}(C + \tfrac{1}{4})\chi(x)(\mathrm{d}_+ f - \mathrm{d}_- f). \tag{5.18}$$

In the next section we make use of the following brackets:

$$\{\psi(x), \psi(y)\} = \tfrac{1}{4}\psi(x)\psi(y)\,\mathrm{sign}(x-y) \tag{5.19}$$

$$\{\chi(x), \chi(y)\} = \tfrac{1}{4}\chi(x)\chi(y)\,\mathrm{sign}(x-y) \tag{5.20}$$

$$\{\psi(x), \chi(y)\} = -\tfrac{1}{4}\psi(x)\chi(y)\,\mathrm{sign}(x-y) + \chi(x)\psi(y)\theta(x-y). \tag{5.21}$$

6. Local commutativity and canonicity

Let $f(U_+, U_-, T)$ be a functional defined on the space $\mathscr{M}$ (3.8). There is a natural notion of admissibility for such functional which includes the requirement of existence and commutativity of various 'partial' derivatives. According to the results of the previous section, we have some freedom to choose the notion of admissibility with respect to the first and second variables. For the purposes of this section, it suffices to consider only the functionals admissible in the sense of $\mathcal{O}'' \times \mathcal{O}''$. For such functionals, define the Poisson bracket as follows:

$$\{f, g\} = \{f, g\}_+ - \{f, g\}_- + \mathrm{tr}\left(\frac{\partial f}{\partial T} \otimes \frac{\partial g}{\partial T}\{T \overset{\otimes}{,} T\}\right) \tag{6.1}$$

where $\{\cdot, \cdot\}_\pm$ is the Poisson bracket with respect to $U_\pm$, described in the previous section (values of $C_\pm$ do not matter here)

$$\frac{\partial f}{\partial T} = \begin{pmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{pmatrix} \qquad \text{if } T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $\{T \underset{,}{\otimes} T\}$ is a 4×4 matrix unknown for the present. In (6.1) and later on we use the tensorial notations customary for the inverse scattering transform method [18]. The brackets (5.1)–(5.3) are formal consequences of (6.1).

The following brackets are essentially nothing but a tensorial form of (5.19)–(5.21):

$$\begin{aligned}
&\{\Omega_+(x) \underset{,}{\otimes} \Omega_+(y)\} = \Omega_+(x)\otimes\Omega_+(y)\rho(x-y)\\
&\{\Omega_-(x) \underset{,}{\otimes} \Omega_-(y)\} = -\rho(x-y)\Omega_-(x)\otimes\Omega_-(y)\\
&\{\Omega_+(x) \underset{,}{\otimes} \Omega_-(y)\} = 0 \qquad \{\Omega_\pm(x) \underset{,}{\otimes} T\} = 0
\end{aligned} \tag{6.2}$$

where $\rho(x) = r^{\mathrm{T}}\theta(x) - r\theta(-x)$, the matrix r being of the form

$$r = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 4 & -1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.3}$$

Now we are going to demonstrate that the condition of local commutativity of the fundamental field Φ uniquely determines the bracket $\{T \underset{,}{\otimes} T\}$ missed in (6.2). Using the representation (3.11) for Φ, we obtain

$$\begin{aligned}
&\{\Phi(x^+, x^-), \Phi(y^+, y^-)\}\\
&\quad = \exp(\Phi(x^+, x^-) + \Phi(y^+, y^-)) \times \{\Omega_+(x^+)T\Omega_-(x^-), \Omega_+(y^+)T\Omega_-(y^-)\}\\
&\quad = \exp(\Phi(x^+, x^-) + \Phi(y^+, y^-))\\
&\qquad \times (\{\Omega_+(x^+) \underset{,}{\otimes} \Omega_+(y^+)\} \cdot T\otimes T \cdot \Omega_-(x^-)\otimes\Omega_-(y^-)\\
&\qquad + \Omega_+(x^+)\otimes\Omega_+(y^+) \cdot \{T \underset{,}{\otimes} T\} \cdot \Omega_-(x^-)\otimes\Omega_-(y^-)\\
&\qquad + \Omega_+(x^+)\otimes\Omega_+(y^+) \cdot T\otimes T \cdot \{\Omega_-(x^-) \underset{,}{\otimes} \Omega_-(y^-)\})\\
&\quad = \exp(\Phi(x^+, x^-) + \Phi(y^+, y^-))\Omega_+(x^+)\otimes\Omega_+(y^+)(\{T \underset{,}{\otimes} T\}\\
&\qquad + \rho(x^+ - y^+)T\otimes T - T\otimes T\rho(x^- - y^-))\Omega_-(x^-)\otimes\Omega_-(y^-).
\end{aligned}$$

The condition of this bracket vanishing in the right space-time cone, $x^+ - y^+ > 0, x^- - y^- > 0$, gives

$$\{T \underset{,}{\otimes} T\} = [-r^{\mathrm{T}}, T\otimes T]$$

(the square brackets designate the matrix commutator). The same condition in the left cone, $x^+ - y^+ < 0$, $x^- - y^- < 0$, gives

$$\{T \underset{,}{\otimes} T\} = [r, T\otimes T]. \tag{6.4}$$

Fortunately, both these expressions for $\{T \underset{,}{\otimes} T\}$ are equivalent. The bracket (6.3), (6.4) equips PSL_2 with the structure of the Poisson Lie group [19].

Now let us verify the canonicity of the fundamental field. The bracket $\{\Phi, \Phi\}$ can be represented in the form

$$\{\Phi(x^+, x^-), \Phi(y^+, y^-)\} = A(x^+, x^-; y^+, y^-)(\theta(x^+ - y^+) - \theta(x^- - y^-)).$$

The canonical brackets (1.3) are equivalent to the equality

$$2A(x^+, x^-; x^+, x^-) \equiv 1. \tag{6.5}$$

In our case the quantity A has the following form:

$$A(x^+,x^-;y^+,y^-)$$
$$=\exp(\Phi(x^+,x^-)+\Phi(y^+,y^-))$$
$$\times\,\Omega_+(x^+)T\otimes\Omega_+(y^+)T\cdot(\mathscr{P}-\mathscr{E}/2)\cdot\Omega_-(x^-)\otimes\Omega_-(y^-)$$

where $\mathscr{E}$ is a 4×4 unit matrix, $\mathscr{P}$ is a permutation, i.e.

$$\mathscr{P}=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\Rightarrow\mathscr{P}\Omega_-(x^-)\otimes\Omega_-(y^-)=\Omega_-(y^-)\otimes\Omega_-(x^-)$$

so the condition (6.5) can be easily verified:

$$2A(x^+,x^-;x^+,x^-)=\exp(2\Phi(x^+,x^-))(\Omega_+(x^+)T\Omega_-(x^-))^2\equiv 1.$$

While speaking about the canonical brackets, note should be taken that in the case of singular fields neither π, φ themselves nor their smoothed versions are admissible functionals. Indeed, in the singular case the field $\varphi(x)$ possesses integrable singularities at some points x_j (see (2.5)), therefore, we can legitimately consider the smoothed field

$$\varphi(\xi)=\int dx\,\xi(x)\varphi(x)\qquad \xi\in S(\mathbb{R}).$$

The functional $\varphi(\xi)$ is differentiable with respect to $U_\pm$, but, contrary to the definition of admissible functional, its variational derivatives are not smooth, e.g. in a neighbourhood of the point x_j we have

$$\frac{\delta\varphi(\xi)}{\delta U_+(y)}\sim(y-x_j)^2\log|y-x_j|.$$

The field $\pi(x)$ possesses non-integrable singularities at the same points x_j, therefore, we are even unable to define its smoothed version without further assumptions. Let us try the principal value regularization:

$$\pi(\xi)=\text{v.p.}\int dx\,\xi(x)\pi(x)\qquad \xi\in S(\mathbb{R}).$$

The functional $\pi(\xi)$, thus defined, is differentiable with respect to $U_\pm$, but its variational derivatives are not smooth, e.g. in a neighbourhood of the point x_j we have

$$\frac{\delta\pi(\xi)}{\delta U_+(y)}\sim(y-x_j)\log|y-x_j|.$$

However, the fundamental field $\Phi(t,x)$, after smoothing with respect to both variables, appears to be an admissible functional (Φ possesses integrable singularities).

7. Hamiltonian representation of the conformal algebra

For $X=X^+\oplus X^-\in\mathscr{A}_*$, define

$$Q_X=\int dx\,(X^+(x)U_+(x)-X^-(x)U_-(x)).$$

Using (5.17), (5.18) and (6.1), one can verify that Q_X properly generates the action of $\mathscr{A}_*$ on $\psi_\pm(x)$, $\chi_\pm(x)$, T, and consequently on the fundamental field Φ (3.11). Further,

$$\{Q_X, Q_Y\} = -Q_{[X, Y]} + \Delta(X, Y) \tag{7.1}$$

where $\Delta(X, Y) = \frac{1}{2}(\omega(X^+, Y^+) - \omega(X^-, Y^-))$ (ω is given by (4.2)). Thus, the mapping $X \to -Q_X$ is a Hamiltonian representation (in the algebra $\mathcal{O}'' \times \mathcal{O}''$) of the conformal algebra $\mathscr{A}_*$ centrally extended by means of the 2-cocycle Δ.

The Poisson structure (6.3), (6.4) on PSL_2 is degenerate, and so is the total Poisson structure (6.1). In so far as the latter is non-degenerate on $M_{n_\pm}$'s, its symplectic leaves are determined by those of PSL_2. The symplectic leaves of PSL_2 are described as follows. The condition $b=c=0$ defines a subset of PSL_2 which is a union of the zero-dimensional leaves labelled by $a>0$. For other leaves, $b^2+c^2 \neq 0$, and the condition $\varkappa = c/b =$ constant distinguishes one (if $-\infty < \varkappa < 0$) or two (if $0 \leqslant \varkappa \leqslant \infty$) two-dimensional leaves. They are labelled by $\varkappa$ and by the sign of a if $0 \leqslant \varkappa \leqslant \infty$ (we assume that $b \geqslant 0$; if $b=0$, then $c \geqslant 0$; if $b=c=0$, then $a>0$).

The bracket (6.1) cannot generate the action of $\mathscr{A}$ on T (4.7), so let us consider the algebra of admissible functionals $\mathcal{O}' \times \mathcal{O}'$ and generalize the bracket (6.1) as follows:

$$\begin{aligned}\{f, g\} = \{f, g\}_+ - \{f, g\}_- + \operatorname{tr}\left(\frac{\partial f}{\partial T} \otimes \frac{\partial g}{\partial T}\{T \underset{,}{\otimes} T\}\right)\\ + \operatorname{tr}(\hat{f} \otimes \hat{g}\Pi) + \operatorname{tr}\Xi\left(\hat{f} \otimes \frac{\partial g}{\partial T} - \hat{g} \otimes \frac{\partial f}{\partial T}\right)\end{aligned} \tag{7.2}$$

where $\{\cdot, \cdot\}_\pm$ has the same meaning as in (6.1), but the constants $C_\pm$ are now equal to $\frac{1}{4}$; the matrix $\{T \underset{,}{\otimes} T\}$ is given by (6.3), (6.4);

$$\Pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p & u & v & q \\ -p & -v & -u & -q \\ 0 & s & -s & 0 \end{pmatrix} \tag{7.3}$$

$$\Xi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes (h_1 \sigma T + h_2 T \sigma) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (k_1 \sigma T + k_2 T \sigma) \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.4}$$

(the non-dynamical real parameters $p, u, v, q, s, h_1, h_2, k_1, k_2$ appearing in (7.3), (7.4), will be specified later); and lastly the 'cap'-operation is defined as follows:

$$\hat{f} = \begin{pmatrix} D_+^+ f & D_-^+ f \\ D_+^- f & D_-^- f \end{pmatrix} \qquad D_\pm^+ = \mathrm{d}_+^{(+)} \pm \mathrm{d}_-^{(+)} \qquad D_\pm^- = \mathrm{d}_+^{(-)} \pm \mathrm{d}_-^{(-)} \tag{7.5}$$

(the upper '$\pm$' correspond to indices of $U_\pm$). The bracket (7.2)–(7.5) reduces to (6.1) for the functionals from $\mathcal{O}'' \times \mathcal{O}''$.

The bracket (7.2)–(7.5) is well defined on the set $\mathscr{M}$ (3.11), because det T is a central element for any values of parameters. Further, $\{f, \beta_\pm\} = (\text{something}) \times \beta_\pm$ for any functional $f \in \mathcal{O}' \times \mathcal{O}'$. This means that the set $\mathscr{M}$ and its subsets $M_{n_+} \times M_{n_-} \times PSL_2$ are closed against the Hamiltonian flows.

It is quite obvious that the bracket (7.2)–(7.5) is bilinear, antisymmetric and satisfies the Leibniz identity.

Proposition 7.1. The bracket (7.2)–(7.5) satisfies the Jacobi identity in the case of the algebra $\mathcal{O}' \times \mathcal{O}'$ for any values of the parameters $p, u, v, q, s, h_1, h_2, k_1, k_2$.

In contrast to the Jacobi identity, the properties of degeneracy depend on the values of parameters. We shall not discuss all variants but only those which admit the Hamiltonian representation of the conformal algebra $\mathcal{A}$. In these cases the situation is simple.

Note that $\{f, b\} = b \times$ (something) for any functional f, so the condition $b=0$ defines a submanifold of $\mathcal{M}$ invariant against the Hamiltonian flows. Likewise, the condition $c=0$ defines an invariant submanifold. Thus, we are forced to consider four cases separately: (1) $bc \neq 0$, (2) $b=0$, $c=0$, (3) $b\neq 0$, $c=0$, (4) $b=0$, $c\neq 0$. In each case the bracket (7.2)–(7.5) will be non-degenerate for those values of parameters which admit the Hamiltonian representation of the conformal algebra $\mathcal{A}$. Note, however, that none of the sets of parameters can be used to cover more than one case, i.e. the phase spaces that will be obtained cannot be considered as leaves of some Poisson structure on $\mathcal{M}$.

(1) $bc\neq 0$. 'Conformal' choice of parameters: h_1, h_2, k_1 and k_2 are arbitrary except that $(h_1-h_2)^2+(k_1-k_2)^2\neq 0$; $s=0$, $p=\frac{1}{4}(k_1-k_2)W$, $u=-\frac{1}{4}(k_1-k_2)V$, $v=\frac{1}{4}(h_1-h_2)W$, $q=-\frac{1}{4}(h_1-h_2)V$, where we have introduced auxiliary variables V, W:

$$V=\frac{h_1-h_2-(k_1-k_2)\nu}{(h_1-h_2)^2+(k_1-k_2)^2} \qquad W=\frac{k_1-k_2+(h_1-h_2)\nu}{(h_1-h_2)^2+(k_1-k_2)^2} \qquad \nu\in\mathbb{R}$$

(in all, five parameters have left). For $X=X^+\oplus X^-\in\mathcal{A}$, define

$$Q_X=R_X+K_X\log|\beta_+|+L_X\log|\beta_-|+M_X\log|bc|+N_X\log|b/c|$$

where

$$R_X=\int \mathrm{d}x\,(X^+(x)U_+(x)-X^-(x)U_-(x))-\tfrac{1}{2}D(X^+)\log|\beta_+|+\tfrac{1}{2}D(X^-)\log|\beta_-| \tag{7.6}$$

$$\begin{gathered} K_X=\mathrm{tr}\,K^{\mathrm{T}}\begin{pmatrix} S(X^+) & D(X^+)\\ S(X^-) & D(X^-)\end{pmatrix} \qquad K=-\tfrac{1}{8}V\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\\ S(X)=X'(+)+X'(-) \qquad D(X)=X'(+)-X'(-). \end{gathered} \tag{7.7}$$

The quantities L_X, M_X, N_X have the same form as K_X, the corresponding matrices being

$$L=-\tfrac{1}{8}W\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} \qquad M=\tfrac{1}{2}\begin{pmatrix}0 & h_2V+k_2W\\ 0 & h_1V+k_1W\end{pmatrix} \qquad N=-\tfrac{1}{8}\begin{pmatrix}V & 0\\ W & 0\end{pmatrix}.$$

One can verify that

$$\delta_X\beta_\pm=\{Q_X,\beta_\pm\}=\tfrac{1}{2}\beta_\pm(\partial X^\pm(+)+\partial X^\pm(-))$$

$$\delta_X\psi_\pm(x)=\{Q_X,\psi_\pm(x)\}=X^\pm(x)\psi'_\pm(x)-\tfrac{1}{2}\,\partial X^\pm(x)\,\psi_\pm(x)-\tfrac{1}{4}\psi_\pm(x)D(X^\pm)$$

$$\delta_X\chi_\pm(x)=\{Q_X,\chi_\pm(x)\}=X^\pm(x)\chi'_\pm(x)-\tfrac{1}{2}\,\partial X^\pm(x)\chi_\pm(x)+\tfrac{1}{4}\chi_\pm(x)D(X^\pm)$$

$$\delta_X T=\{Q_X,T\}=-\tfrac{1}{4}D(X^+)\sigma T+T\sigma\tfrac{1}{4}D(X^-).$$

These formulae are infinitesimal analogues of (4.4)–(4.7). Thus, Q_X properly generates the action of $\mathcal{A}$ on $\beta_\pm$, $\psi_\pm(x)$, $\chi_\pm(x)$, T, and hence on Φ. Further, Q_X satisfies (7.1) with

$$\Delta(X, Y)=\Delta'(X, Y)-\tfrac{1}{16}(VS(X^+)+WS(X^-))(D(Y^+)+D(Y^-))$$
$$+\tfrac{1}{16}(VS(Y^+)+WS(Y^-))(D(X^+)+D(X^-))$$

where

$$\Delta'(X, Y)=\tfrac{1}{2}(\omega(X^+, Y^+)-\omega(X^-, Y^-))+\tfrac{1}{4}(\tilde{\omega}(X^+, Y^+)-\tilde{\omega}(X^-, Y^-)). \tag{7.8}$$

It is worth noting that $R_X \in \mathcal{O}'' \times \mathcal{O}''$ even for $X \in \mathscr{A}$; further, the functional R_X properly generates the action of the algebra $\mathscr{A}$ on $\beta_\pm$, $\psi_\pm(x)$, $\chi_\pm(x)$, but not on T, with the help of the bracket (6.1), the 2-cocycle being of the form (7.8).

(2) $b=c=0$. In this case h_1, h_2, k_1, k_2 enter into (7.2)–(7.5) only through combinations $h=h_1+h_2$, $k=k_1+k_2$ which can be arbitrary except that $h^2+k^2\neq 0$; $s=0$, $p=\tfrac{1}{4}kW$, $u=-\tfrac{1}{4}kV$, $v=\tfrac{1}{4}hW$, $q=-\tfrac{1}{4}hV$, where

$$V=\frac{h-k\nu}{h^2+k^2} \qquad W=\frac{k+h\nu}{h^2+k^2} \qquad \nu\in\mathbb{R}$$

(in all, three parameters). For $X=X^+\oplus X^-\in\mathscr{A}$, define

$$Q_X=R_X+K_X\log|\beta_+|+L_X\log|\beta_-|+M_X\log|a|$$

where R_X is given by (7.6); K_X, L_X, M_X are of the form (7.7) with

$$K=-\tfrac{1}{8}V\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad L=\tfrac{1}{8}W\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix} \qquad M=-\tfrac{1}{4}\begin{pmatrix}V & 0\\ W & 0\end{pmatrix}$$

$$\Delta(X, Y)=\Delta'(X, Y)-\tfrac{1}{16}(VS(X^+)+WS(X^-))(D(Y^+)-D(Y^-))$$
$$+\tfrac{1}{16}(VS(Y^+)+WS(Y^-))(D(X^+)-D(X^-)).$$

(3) $b>0$, $c=0$. For $X=X^+\oplus X^-\in\mathscr{A}$, define

$$Q_X=R_X+K_X\log|\beta_+|+L_X\log|\beta_-|+M_X\log|a|+N_X\log|b|$$

where R_X is given by (7.6). K_X, L_X, M_X, N_X are of the form (7.7). There are two sets of parameters which admit the Hamiltonian representation of $\mathscr{A}$.

(a) $h_1=h_2=k_1=k_2=u=v=0$, $p=\tfrac{1}{4}$, $q=-\tfrac{1}{4}$, $s\in\mathbb{R}$. In this case

$$K=0 \qquad L=0 \qquad M=\tfrac{1}{2}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} \qquad N=\tfrac{1}{2}\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}$$

$$\Delta(X, Y)=\Delta'(X, Y)-\tfrac{1}{4}(D(X^+)D(Y^-)-D(X^-)D(Y^+)).$$

(b) $p=q=u=v=0$; h_1, h_2, k_1, k_2, and s are arbitrary except that $l\equiv s-4(h_1k_2-h_2k_1)\neq 0$. In this case

$$K=(2l)^{-1}\begin{pmatrix}0 & k_2\\ 1/4 & k_1\end{pmatrix} \qquad L=-(2l)^{-1}\begin{pmatrix}1/4 & h_2\\ 0 & h_1\end{pmatrix}$$

$$M=(2l)^{-1}\begin{pmatrix}-k_1+k_2 & s\\ h_1-h_2 & s\end{pmatrix} \qquad N=(2l)^{-1}\begin{pmatrix}k_1+k_2 & -s\\ -h_1-h_2 & s\end{pmatrix}$$

$$\Delta(X, Y)=\Delta'(X, Y)+(4l)^{-1}(\tfrac{1}{4}(S(X^+)S(Y^-)-S(X^-)S(Y^+))+B(X, Y))$$

$$B(X, Y) = -s(D(X^+)D(Y^-) - D(X^-)D(Y^+))$$
$$+ k_1(S(X^+)D(Y^-) - D(X^-)S(Y^+)) + k_2(S(X^+)D(Y^+) - D(X^+)S(Y^+))$$
$$- h_1(S(X^-)D(Y^-) - D(X^-)S(Y^-)) - h_2(S(X^-)D(Y^+) - D(X^+)S(Y^-)).$$

(4) $b=0$, $c>0$. For $X = X^+ \oplus X^- \in \mathscr{A}$, define

$$Q_X = R_X + K_X \log|\beta_+| + L_X \log|\beta_-| + M_X \log|a| + N_X \log|c|$$

where R_X is given by (7.6), K_X, L_X, M_X, N_X are of the form (7.7). There are again two sets of parameters which admit the Hamiltonian representation of $\mathscr{A}$.

(a) $h_1 = h_2 = k_1 = k_2 = u = v = 0, p = \frac{1}{4}$, $q = -\frac{1}{4}$, $s \in \mathbb{R}$. In this case

$$K = 0 \qquad L = 0 \qquad M = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad N = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\Delta(X, Y) = \Delta'(X, Y) + \tfrac{1}{4}(D(X^+)D(Y^-) - D(X^-)D(Y^+)).$$

(b) $p = q = u = v = 0$; h_1, h_2, k_1, k_2, and s are arbitrary except that $m \equiv s + 4(h_1k_2 - h_2k_1) \neq 0$. In this case

$$K = (2m)^{-1}\begin{pmatrix} 0 & -k_2 \\ 1/4 & -k_1 \end{pmatrix} \qquad L = (2m)^{-1}\begin{pmatrix} -1/4 & h_2 \\ 0 & h_1 \end{pmatrix}$$

$$M = -(2m)^{-1}\begin{pmatrix} -k_1 + k_2 & s \\ h_1 - h_2 & s \end{pmatrix} \qquad N = (2m)^{-1}\begin{pmatrix} k_1 + k_2 & -s \\ -h_1 - h_2 & s \end{pmatrix}$$

$$\Delta(X, Y) = \Delta'(X, Y) + (4m)^{-1}\left(\tfrac{1}{4}(S(X^+)S(Y^-) - S(X^-)S(Y^+)) - B(X, Y)\right).$$

If $X^+(x) \equiv 1$, $X^-(x) \equiv -1$, then Q_X is nothing but the Hamiltonian of the model. In all cases considered in this section it looks as follows (note the difference from [7, 8]):

$$H = \int dx(U_+(x) + U_-(x))$$
$$= \int dx\left(\tfrac{1}{2}\pi^2(x) + \tfrac{1}{2}(\varphi'(x))^2 + 2\,e^{2\varphi(x)} - \varphi''(x)\right) \tag{7.9}$$

Using spectral representation for the first integral, i.e. trace identity [18, 20], one can prove that H is positive only on the phase spaces which are subsets of $M_0 \times M_0 \times PSL_2$. The fields which constitute such phase spaces may possess one, two or no points of singularity (see proposition 3.3). For non-singular fields, the positiveness of the Hamiltonian is obvious from (1.4) that in this case is equivalent to (7.9).

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Appendix

Here the expressions for the constants appearing in formulae (3.4) are listed.

(a) $b=0,\ c=0 \Leftrightarrow$ (3.4*a*) with $u_{\pm}=\frac{1}{2}\log|a^{\pm 2}\beta_-/\beta_+|$,

$$q_+=-\frac{(-1)^{n_-}\alpha_-a^2+(-1)^{n_+}\alpha_+}{|\beta_-|a^2+|\beta_+|}\qquad q_-=\frac{(-1)^{n_-}\vartheta_-+(-1)^{n_+}\vartheta_+a^2}{|\beta_-|+|\beta_+|a^2}$$

(b) $b=0,\ c\neq 0 \Leftrightarrow$ (3.4*b*) with $u_-=\frac{1}{2}\log|a^{-2}\beta_-/\beta_+|$,

$$q_-=\frac{(-1)^{n_-}\vartheta_-+(-1)^{n_+}\vartheta_+a^2+ac}{|\beta_-|+|\beta_+|a^2}\qquad B_+^2=c^2|\beta_+\beta_-|$$

$$q_++\tau_+=-\frac{(-1)^{n_+}\alpha_++a/c}{|\beta_+|}\qquad q_+-\tau_+=-\frac{(-1)^{n_-}\alpha_-+1/(ac)}{|\beta_-|}$$

(c) $b\neq 0,\ c=0 \Leftrightarrow$ (3.4*c*) with $B_-^2=b^2|\beta_+\beta_-|$,

$$q_-+\tau_-=\frac{(-1)^{n_+}\vartheta_++1/(ab)}{|\beta_+|}\qquad q_--\tau_-=\frac{(-1)^{n_-}\vartheta_-+a/b}{|\beta_-|}$$

$$q_+=-\frac{(-1)^{n_-}\alpha_-a^2+(-1)^{n_+}\alpha_++ab}{|\beta_-|a^2+|\beta_+|}\qquad u_+=\frac{1}{2}\log|a^2\beta_-/\beta_+|$$

(d) $b\neq 0,\ c\neq 0 \Leftrightarrow$ (3.4*d*) with $B_-^2=b^2|\beta_+\beta_-|$, $B_+^2=c^2|\beta_+\beta_-|$,

$$q_-+\tau_-=\frac{(-1)^{n_+}\vartheta_++d/b}{|\beta_+|}\qquad q_--\tau_-=\frac{(-1)^{n_-}\vartheta_-+a/b}{|\beta_-|}$$

$$q_++\tau_+=-\frac{(-1)^{n_+}\alpha_++a/c}{|\beta_+|}\qquad q_+-\tau_+=-\frac{(-1)^{n_-}\alpha_-+d/c}{|\beta_-|}.$$

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